MATH 2050C Lecture 10 (Feb 16)

[Problem set 5 posted, due on Feb 24.]

Last time: "Limit Thm" ASSUME $\lim(x_n), \lim(y_n)$ exist.

$$\begin{cases} \lim(x_n \pm y_n) = \lim(x_n) \pm \lim(y_n) \\ \lim(x_n y_n) = \lim(x_n) \cdot \lim(y_n) \\ \lim\left(\dfrac{x_n}{y_n}\right)_{\neq 0} = \dfrac{\lim(x_n)}{\lim(y_n) \neq 0} \end{cases}$$

$$\begin{cases} \text{If } x_n \leq y_n \quad \forall n \in \mathbb{N} \\ \text{then } \lim(x_n) \leq \lim(y_n) \end{cases}$$

also OK. $\forall n \geq K$ for some $K \in \mathbb{N}$

Q: How to prove that $\lim(x_n)$ exist?

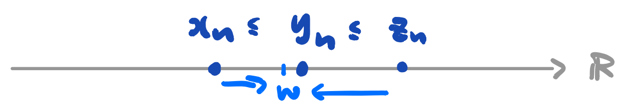
Thm: ("Squeeze / Sandwich Theorem")

Let $(x_n), (y_n), (z_n)$ be seq. of real numbers s.t.

(1) $x_n \leq y_n \leq z_n \quad \forall n \in \mathbb{N}$ ($\forall n \geq K$ for some K)

(2) $\lim(x_n) = w = \lim(z_n)$

THEN, $\lim(y_n) = w$.



Remark: We do NOT need to assume $\lim(y_n)$ exists, it follows from the theorem.

E.g.) $\lim\left(\dfrac{\sin n}{n}\right) = 0$ because $\overset{0}{\uparrow}\!-\dfrac{1}{n} \leq \overset{0}{\uparrow}\!\dfrac{\sin n}{n} \leq \overset{0}{\uparrow}\!\dfrac{1}{n}$

Proof: Let $\varepsilon > 0$ be fixed but arbitrary.

$\lim(x_n) = w \Rightarrow \exists K_1 \in \mathbb{N}$ s.t. $|x_n - w| < \varepsilon \quad \forall n \geq K_1$ —— (*)

$\lim(z_n) = w \Rightarrow \exists K_2 \in \mathbb{N}$ s.t. $|z_n - w| < \varepsilon \quad \forall n \geq K_2$ —— (**)

Choose $K := \max\{K_1, K_2\}$, then $\forall n \geq K$

$$-\varepsilon \overset{(*)}{<} x_n - w \overset{(1)}{\leq} y_n - w \overset{(1)}{\leq} z_n - w \overset{(**)}{<} \varepsilon$$

i.e. $|y_n - w| < \varepsilon$

$\square$

Thm: ("Ratio Test")

Let (x_n) be a seq. s.t.

(1) $x_n > 0 \quad \forall n \in \mathbb{N}$

(2) $\lim\left(\frac{x_{n+1}}{x_n}\right) = L < 1$

crucial!

THEN, $\lim(x_n) = 0$.

Motivation

Geometric seq.

$(ar^n) \to 0$

provided $|r| < 1$

Ex: Prove this!

E.g.) Consider $(x_n) = \left(\frac{n}{2^n}\right)$, then

$$\left(\frac{x_{n+1}}{x_n}\right) = \left(\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right) = \left(\frac{n+1}{n} \cdot \frac{1}{2}\right) \to \frac{1}{2} < 1$$

By Ratio Test, $\lim\left(\frac{n}{2^n}\right) = 0$.

Proof: Idea: Compare (x_n) with a geometric seq. (b^n), where $0<b<1$ and apply Squeeze Thm!

$\frac{x_{n+1}}{x_n} \approx L < 1$

$x_{n+1} \approx L x_n$

$x_{n+2} \approx L^2 x_n$

$x_{n+k} \approx L^k x_n$

Since $L < 1$, $\exists r \in \mathbb{R}$ s.t. $L < r < 1$.

Take $\varepsilon = r - L > 0$, by (2), $\exists K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon = r - L \quad \forall n \geq K$$

$$\Rightarrow \quad 0 \overset{(1)}{<} \frac{x_{n+1}}{x_n} < L + (r - L) = r < 1$$

Thus, $x_{n+1} < r x_n \quad \forall\, n \geq K$.

i.e $\quad 0 < x_n < r x_{n-1} < r^2 x_{n-2} < \cdots < r^{n-K} x_K$

Note: $\lim\limits_{n\to\infty} (r^{n-K} x_K) = 0$ since $r < 1$.

K fixed

By Sandwich Thm, $\lim (x_n) = 0$

Remark: Ratio Test fails if $L = 1$.

Consider the seq. $(x_n) = (n)$, which is divergent

But $\left(\frac{x_{n+1}}{x_n}\right) = \left(\frac{n+1}{n}\right) \to 1 = 1$

Ex: Construct an example that $\left(\frac{x_{n+1}}{x_n}\right) \to 1$ from below.